

THEOREMS ON TWIN PRIMES - DUAL CASE

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ABSTRACT. For the sequence: $\tilde{c}(1) = 2$ and for $n \geq 2$,

$$\tilde{c}(n) = \tilde{c}(n-1) + \begin{cases} \gcd(n, \tilde{c}(n-1)), & \text{if } n \text{ is odd} \\ \gcd(n-2, \tilde{c}(n-1)), & \text{if } n \text{ is even,} \end{cases}$$

which is "dual" to earlier considered sequence

$$c(n) = c(n-1) + \begin{cases} \gcd(n, c(n-1)), & \text{if } n \text{ is even} \\ \gcd(n-2, c(n-1)), & \text{if } n \text{ is odd,} \end{cases}$$

we prove similar theorems of its connection with twin primes. Besides, we consider several new sequences of such type which are also connected with twin primes. Finally, with help of a postulate we prove the infinity of twin primes.

1. INTRODUCTION

In [2] we posed, in particular, the following conjecture

Conjecture 1. *Let $\tilde{c}(1) = 2$ and for $n \geq 2$,*

$$\tilde{c}(n) = \tilde{c}(n-1) + \begin{cases} \gcd(n, \tilde{c}(n-1)), & \text{if } n \text{ is odd} \\ \gcd(n-2, \tilde{c}(n-1)), & \text{if } n \text{ is even.} \end{cases}$$

Then every record (more than 3) of the values of difference $\tilde{c}(n) - \tilde{c}(n-1)$ is greater of twin primes.

The first records are (cf. sequence A167495 in [5])

$$(1.1) \quad 5, 13, 31, 61, 139, 283, 571, 1153, 2311, 4651, 9343, 19141, 38569, \dots$$

We use the same way as in our paper [4] which is devoted to study a sequence dual to the now considered one. Our observations of the behavior of sequence $\{\tilde{c}(n)\}$ are the following:

1) In some sequence of arguments $\{m_i\}$ we have $\frac{\tilde{c}(m_i)-3}{m_i-3} = 3/2$. These values of arguments we call *the fundamental points*. The first fundamental point are

$$7, 27, 63, 123, 279, 567, 1143, 2307, 4623, 9303, 18687, \dots$$

2) For every two adjacent fundamental points $m_j < m_{j+1}$, we have $m_{j+1} \geq 2m_j - 3$.

- 3) For $i \geq 2$, the numbers $\frac{m_i-5}{2}$, $\frac{m_i-1}{2}$ are twin primes (and, consequently, $m_i \equiv 3 \pmod{12}$).
- 4) In points $m_i + 1$ we have $\tilde{c}(m_i + 1) - \tilde{c}(m_i) = \frac{m_i-1}{2}$. These increments we call *the main increments* of sequence $\{\tilde{c}(n)\}$, while other nontrivial (i.e. more than 1) increments we call *the minor increments*.
- 5) For $i \geq 2$, denote h_i the number of minor increments between adjacent fundamental points m_i and m_{i+1} and T_i the sum of these increments. Then $T_i \equiv h_i \pmod{6}$.
- 6) For $i \geq 2$, the minor increments between adjacent fundamental points m_i and m_{i+1} could occur only before $m_{i+1} - \sqrt{2(m_{i+1} - 1)} - 2$.

Below we show that the validity of all these observations follow only from 6).

Theorem 1. *If observation 6) is true then observation 1)-5) are true as well.*

Corollary 1. *If 1) observation 6) is true and 2) the sequence $\{\tilde{c}(n)\}$ contains infinitely many fundamental points, then there exist infinitely many twin primes.*

Besides, in connection with Conjecture 1 we think that

Conjecture 2. *For $n \geq 16$, the main and only main increments are the record differences $\tilde{c}(n) - \tilde{c}(n-1)$.*

2. PROOF OF THEOREM 1

We use induction. Suppose $n_1 \geq 28$ is a number of the form $12l+4$ (for $n_1 < 28$ the all observations are verified directly). Let n_1-1 is a fundamental point and for $n := \frac{n_1-4}{2}$, $n \mp 1$ are twin primes. Thus

$$\tilde{c}(n_1 - 1) = \frac{3}{2}(n_1 - 4) + 3 = \frac{3}{2}n_1 - 3.$$

Since n_1 is even and

$$\gcd\left(\frac{3}{2}n_1 - 3, n_1 - 2\right) = \frac{n_1}{2} - 1,$$

then we have a main increment such that

$$(2.1) \quad \tilde{c}(n_1) = 2n_1 - 4.$$

Here we distinguish two cases:

A) Up to the following fundamental point there are only trivial increments. The inductive step in this case we formulate as the following.

Theorem 2. *If $27 \leq m_i < m_{i+1}$ are adjacent fundamental points without minor increments between them, then i) $m_{i+1} = 2m_i - 3$;
ii) If $\frac{m_i-5}{2}, \frac{m_i-1}{2}$ are twin primes, then $\frac{m_{i+1}-5}{2}, \frac{m_{i+1}-1}{2}$ are twin primes as well.*

Note that really, for the first time, Case **A)** appears for $m_3 = 63$, such that, by Theorem 2, we have two pairs of twin primes: (29,31), (59,61).

Inductive step in case A)

Continuing (2.1), we have

$$\tilde{c}(n_1 + 1) = 2n_1 - 3,$$

$$\tilde{c}(n_1 + 2) = 2n_1 - 2,$$

...

$$\tilde{c}(2n_1 - 5) = 3n_1 - 9,$$

Since $\frac{3n_1-12}{2n_1-8} = 3/2$, then we conclude that $2n - 1 - 5$ is the second fundamental point in the inductive step. By the definition of the sequence, denoting $n_2 = 2n_1 - 4$, we have

$$(2.2) \quad \tilde{c}(n_2) = 2n_2 - 4.$$

Note that, since $n_1 = 12l + 4$, then $n_2 = 12l_1 + 4$, where $l_1 = 2l$.

Furthermore, from the run of formulas (2.2) we find for $3 \leq j \leq \frac{n_1-2}{2}$:

$$\tilde{c}(2n_1 - 2j - 1) = 3n_1 - 2j - 5,$$

$$\tilde{c}(2n_1 - 2j) = 3n_1 - 2j - 4.$$

This means that

$$\gcd(2n_1 - 2j - 2, 3n_1 - 2j - 5) = 1, \text{ i.e. } \gcd(j - 2, n_1 - 3) = 1.$$

Note that, for the considered values of n_1 we have $\frac{n_1-2}{2} \geq \sqrt{n_1-3}$, then $n_1 - 3 = \frac{n_2-2}{2}$ is prime.

On the other hand,

$$\tilde{c}(2n_1 - 2j) = 3n_1 - 2j - 4,$$

$$\tilde{c}(2n_1 - 2j + 1) = 3n_1 - 2j - 3.$$

Thus, for $7 \leq j \leq \frac{n_1-2}{2}$,

$$\gcd(2n_1 - 2j + 1, 3n_1 - 2j - 4) = 1, \text{ i.e. } \gcd(2j - 11, n_1 - 5) = 1.$$

Here, for the considered values of n_1 we also have $2n_1 - 13 \geq \sqrt{n_1 - 5}$, then $n_1 - 5 = \frac{n_2-6}{2}$ is prime. ■

B) Up to the following fundamental point we have some minor increments.

The inductive step we formulate as following.

Theorem 3. *Let observation 6) be true. If $7 \leq m_i < m_{i+1}$ are adjacent fundamental points with a finite number of minor increments between them, then*

i) $m_{i+1} \geq 2m_i$;

ii) *If $\frac{m_i-5}{2}, \frac{m_i-1}{2}$ are twin primes, then $\frac{m_{i+1}-5}{2}, \frac{m_{i+1}-1}{2}$ are twin primes as well.*

Thus the observation 2) will be proved in frameworks of the induction.

Inductive step in case B)

Let in the points $n_1 + l_j$ $j = 1, \dots, h$, before the second fundamental point we have the minor increments t_j , $j = 1, \dots, h$. We have (starting with the first fundamental point $n_1 - 1$)

$$\tilde{c}(n_1 - 1) = \frac{3}{2}n_1 - 3,$$

$$\tilde{c}(n_1) = 2n_1 - 4,$$

$$\tilde{c}(n_1 + 1) = 2n_1 - 3,$$

...

$$\tilde{c}(n_1 + l_1 - 1) = 2n_1 + l_1 - 5.$$

$$(2.3) \quad \tilde{c}(n_1 + l_1) = 2n_1 + l_1 + t_1 - 5,$$

$$\tilde{c}(n_1 + l_1 + 1) = 2n_1 + l_1 + t_1 - 4,$$

...

$$\tilde{c}(n_1 + l_2 - 1) = 2n_1 + l_2 + t_1 - 6,$$

$$(2.4) \quad \tilde{c}(n_1 + l_2) = 2n_1 + l_2 + t_1 + t_2 - 6,$$

...

$$\tilde{c}(n_1 + l_h - 1) = 2n_1 + l_h + t_1 + \dots + t_{h-1} - h - 5,$$

$$\begin{aligned}
(2.5) \quad & \tilde{c}(n_1 + l_h) = 2n_1 + l_h + t_1 + \dots + t_h - h - 4, \\
& \tilde{c}(n_1 + l_h + 1) = 2n_1 + l_h + t_1 + \dots + t_h - h - 3, \\
& \dots
\end{aligned}$$

$$(2.6) \quad \tilde{c}(2n_1 + 2T_h - 2h - 5) = 3n_1 + 3T_h - 3h - 9,$$

where

$$(2.7) \quad T_h = t_1 + \dots + t_h.$$

It is easy to see that $2n_1 + 2T_h - 2h - 5$ is the second fundamental point in the inductive step. Furthermore, subtracting 2 from the even number $2n_1 + 2T_h - 2h - 4$, we see that

$$\gcd(2n_1 + 2T_h - 2h - 6, 3n_1 + 3T_h - 3h - 9) = n_1 + T_h - h - 3.$$

Thus in the point $n_2 := 2n_1 + 2T_h - 2h - 4$ we have the second main increment (in framework of the inductive step):

$$(2.8) \quad \tilde{c}(2n_1 + 2T_h - 2h - 4) = 4n_1 + 4T_h - 4h - 12.$$

Note that, for $n \geq 2$, we have $\tilde{c}(n) \equiv n \pmod{2}$. Therefore, $T_h \geq 3h$ and for the second fundamental point $n_2 - 1 = 2n_1 + 2T_h - 2h - 5$ we find

$$(2.9) \quad n_2 - 1 \geq 2(n_1 - 1) + 4h - 3.$$

This in frameworks of the induction confirms observation 2).

Now, in order to finish the induction, we prove the primality of numbers $\frac{n_2-6}{2} = n_1 + T_h - h - 5$ and $\frac{n_2-2}{2} = n_1 + T_h - h - 3$.

From the run of formulas (2.5)-(2.6) for $7 \leq j \leq \frac{n_1+2T_h-2h-l_h}{2}$ (we cannot cross the upper boundary of the last miner increment) we find

$$\begin{aligned}
& \tilde{c}(2n_1 + 2T_h - 2h - 2j) = 3n_1 + 3T_h - 3h - 2j - 4, \\
& \tilde{c}(2n_1 + 2T_h - 2h - 2j + 1) = 3n_1 + 3T_h - 3h - 2j - 3.
\end{aligned}$$

Thus, for $7 \leq j \leq \frac{n_1+2T_h-2h-l_h}{2}$,

$$\gcd(2n_1 + 2T_h - 2h - 2j + 1, 3n_1 + 3T_h - 3h - 2j - 4) = 1,$$

i.e.

$$\gcd(2j - 11, n_1 + T_h - h - 5) = 1.$$

For the most possible $j = \frac{n_1+2T_h-2h-l_h-1}{2}$ (it is sufficient to consider the case of odd l_h) we should have

$$2j - 11 = n_1 + 2T_h - 2h - l_h - 12 \geq \sqrt{n_1 + T_h - h - 5},$$

or, since $n_2 = 2n_1 + 2T_h - 2h - 4$, then we should have $n_2 - n_1 - l_h - 8 \geq \sqrt{\frac{n_2-6}{2}}$, i.e.

$$(2.10) \quad n_1 + l_h \leq n_2 - \sqrt{\frac{n_2-6}{2}} - 8,$$

Since $n_2 \geq 28$, then this condition, evidently, follows from observation 6) which is written in terms of the fundamental points $m_i = n_i - 1$. Thus from observation 6) we indeed obtain the primality of $\frac{n_2-6}{2} = n_1 + T_h - h - 5$. Furthermore,

$$\tilde{c}(2n_1 + 2T_h - 2h - 2j + 1) = 3n_1 + 3T_h - 3h - 2j - 3,$$

$$\tilde{c}(2n_1 + 2T_h - 2h - 2j + 2) = 3n_1 + 3T_h - 3h - 2j - 2.$$

Thus, for $6 \leq j \leq \frac{n_1+2T_h-2h-l_h}{2}$,

$$\gcd(2n_1 + 2T_h - 2h - 2j, 3n_1 + 3T_h - 3h - 2j - 3) = 1,$$

i.e.

$$\gcd(j - 3, n_1 + T_h - h - 3) = 1.$$

For the most possible $j = \frac{n_1+2T_h-2h-l_h-1}{2}$ (here again sufficiently to consider the case of odd l_h) we should have

$$\frac{n_1 + 2T_h - 2h - l_h - 1}{2} - 3 \geq \sqrt{\frac{n_2-2}{2}},$$

or

$$(2.11) \quad n_1 + l_h \leq n_2 - \sqrt{2(n_2-2)} - 3.$$

This coincides with observation 6). Thus $\frac{n_2-2}{2}$ is prime as well. This completes proof of Theorem 1 ■

Note that in [4] we used the Rowland method [1] to obtain an independent from observation 6) proof of the primality of the greater number. Here we give a parallel proofs for both of numbers.

Corollary 2. *If $p_1 < p_2$ are consecutive seconds of twin primes giving by Theorem 1, then $p_2 \geq 2p_1 - 1$.*

Proof. The corollary easily follows from (2.9). ■

Corollary 3.

$$T_h \equiv h \pmod{6}.$$

Proof. The corollary follows from the well known fact that the half-sum of twin primes not less than 5 is a multiple of 6. Therefore, $n_1 + T_h - h - 4 \equiv 0 \pmod{6}$. Since, by the condition, $n_1 \equiv 4 \pmod{12}$, then we obtain the corollary. ■

Now the observation 5) follows in the frameworks of the induction. The same we can say about observation 4). The observed weak excesses of the exact estimate of Corollary 2 indicate to the smallness of T_h and confirm, by Theorem 1, Conjecture 1.

3. A RULE FOR CONSTRUCTING A PAIR OF TWIN PRIMES $p, p + 2$ BY A GIVEN INTEGER $m \geq 4$ SUCH THAT $p + 2 \geq m$

One can consider a simple rule for constructing a pair of twin primes $p, p + 2$ by a given integer $m \geq 4$ such that $p + 2 \geq m$ quite similar to one over sequence $\{c(n)\}$ (see Section 6 in [4]). To this aim, with m we associate the sequence

$$\begin{aligned} \tilde{c}^{(m)}(1) &= m; \text{ for } n \geq 2, \\ (3.1) \quad \tilde{c}^{(m)}(n) &= \tilde{c}^{(m)}(n-1) + \begin{cases} \gcd(n, \tilde{c}^{(m)}(n-1)), & \text{if } n \text{ is even} \\ \gcd(n-2, \tilde{c}^{(m)}(n-1)), & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus for every m this sequence has the the same formula that the considered one but with another initial condition. Our observation is the following.

Conjecture 3. *Let n^* , where $n^* = n^*(m)$, be point of the last nontrivial increment of $\{\tilde{c}^{(m)}(n)\}$ on the set $A_m = \{1, \dots, m-3\}$ and $n^* = 1$, if there is not any nontrivial increment on A_m . Then numbers $\tilde{c}^{(m)}(n^*) - n^* \mp 1$ are twin primes.*

Evidently, $\tilde{c}^{(m)}(n^*) - n^* + 1 \geq m$ and the equality holds if and only if $n^* = 1$.

The following examples show that, for the same m , the pair of twin primes which is obtained by the considered rule, generally speaking, differs from one which is obtained by the corresponding rule in [4].

Example 1. *Let $m = 577$. Then $n^* = 51$ and $\tilde{c}^{(m)}(n^*) = 669$. Thus numbers $669 - 51 \mp 1$ are twin primes (617, 619), while by the rule in [4] we had another pair: (881, 883).*

Example 2. *Let $m = 3111$. Then $n^* = 123$ and $\tilde{c}^{(m)}(n^*) = 3513$. Thus numbers $3513 - 123 \mp 1$ are twin primes (3389, 3391), while by the rule in [4] we have another pair: (3119, 3121).*

The case of $n^* = 1$ we formulate as the following criterion, which is proved quite similar to Criterion 1 [4].

Criterion 1. *A positive integer $m > 3$ is a greater of twin primes if and only if all the points $1, \dots, m-3$ are points of trivial increments of sequence $\{\tilde{c}^{(m)}(n)\}$.*

4. A NEW SEQUENCE AND AN ASTONISHING OBSERVATION

Consider the sequence which is defined by the recursion:

$$f(1) = 2 \text{ and, for } n \geq 2, \\ f(n) = f(n-1) + \begin{cases} \gcd(n, f(n-1) + 2), & \text{if } n \text{ is even} \\ \gcd(n, f(n-1)), & \text{if } n \text{ is odd.} \end{cases}$$

Here the even points $m_i \neq 8$ in which $f(m_i)/m_i = 3/2$ we call the fundamental points. The increments $\frac{m_i+2}{2}$ in the points $n_i = m_i + 2$ are called main increments and other nontrivial (i.e. different from 1) increments we call miner increments. This sequence also could be studied by method of [4]. It is easy to verify that the nontrivial increments of this sequence differs from ones of the above considered sequence $\{\tilde{c}(n)\}$. But, our observations show that a very astonishing fact, probably, is true: *all records more than 7 for sequences $\{\tilde{c}(n)\}$ and $\{f(n)\}$ coincide!* We think that it is a deep open problem.

5. SOME OTHER NEW SEQUENCES CONNECTED WITH TWIN PRIMES

Here we present three additional new sequences of the considered type, the records of which are undoubtedly connected with twin primes.

1)

$$g(1) = 2 \text{ and, for } n \geq 2, \\ g(n) = g(n-1) + \begin{cases} \gcd(n, g(n-1) + 2), & \text{if } n \text{ is even} \\ \gcd(n-2, g(n-1) + 2), & \text{if } n \text{ is odd.} \end{cases}$$

2)

$$h(1) = 2 \text{ and, for } n \geq 2, \\ h(n) = h(n-1) + \begin{cases} \gcd(n-2, h(n-1) + 2), & \text{if } n \text{ is even} \\ \gcd(n, h(n-1) + 2), & \text{if } n \text{ is odd.} \end{cases}$$

3)

$$i(1) = 2 \text{ and, for } n \geq 2, \\ i(n) = i(n-1) + \gcd(n, i(n-1) + 2(-1)^n).$$

Note that, all records of the second sequence are, probably, the firsts of twin primes.

6. A THEOREM ON TWIN PRIMES WHICH IS INDEPENDENT ON
OBSERVATION OF TYPE 6)

Here we present a new sequence $\{\tilde{a}(n)\}$ with the quite analogous definition of fundamental and miner points for which Corollary 1 is true in a stronger formulation. Using a construction close to those ones that we considered in [3], consider the sequence defined as the following: $\tilde{a}(22) = 40$ and for $n \geq 23$,

$$(6.1) \quad \tilde{a}(n) = \begin{cases} \tilde{a}(n-1) + 1, & \text{if } \gcd(n - (-1)^n - 1, \tilde{a}(n-1)) = 1; \\ 2(n-2) & \text{otherwise} \end{cases}.$$

The sequence has the following first nontrivial differences

$$19, 6, 2, 43, \dots$$

Definition 1. A point m_i is called a fundamental point of sequence (6.1), if it has the form $m_i = 12t + 3$ and $\tilde{a}(m_i) - 3 = \frac{3}{2}(m_i - 3)$. The increments in the points $m_i + 1$ we call the main increments. Other nontrivial increments we call miner increments.

The first two fundamental points of sequence (6.1) are 39 and 87.

Theorem 4. If the sequence $\{\tilde{a}(n)\}$ contains infinitely many fundamental points, then there exist infinitely many twin primes.

Proof. We use induction. Suppose, for some $i \geq 1$, the numbers $\frac{m_i - 3}{2} \mp 1$ are twin primes. Put $n_i = m_i + 1$. Then $n_i \equiv 4 \pmod{12}$ and we have

$$\begin{aligned} \tilde{a}(n_i - 1) &= \frac{3}{2}n_i - 3, \\ \tilde{a}(n_i) &= 2n_i - 4, \end{aligned}$$

We see that the main increment is $\frac{n_i - 2}{2}$. By the condition, before m_{i+1} we can have only a finite set of miner increments. Suppose that, they are in the points $n_i + l_j, j = 1, \dots, h_i$. Then, by (6.1), we have

$$\tilde{a}(n_i + 1) = 2n_i - 3,$$

...

$$\tilde{a}(n_i + l_1 - 1) = 2n_i + l_1 - 5,$$

$$\tilde{a}(n_i + l_1) = 2n_i + 2l_1 - 4,$$

...

$$\tilde{a}(n_i + l_2 - 1) = 2n_i + l_1 + l_2 - 5,$$

$$\tilde{a}(n_i + l_2) = 2n_i + 2l_2 - 4,$$

...

$$\begin{aligned}
& \tilde{a}(n_i + l_h - 1) = 2n_i + l_{h-1} + l_h - 5, \\
(6.2) \quad & \tilde{a}(n_i + l_h) = 2n_i + 2l_h - 4,
\end{aligned}$$

...

$$(6.3) \quad \tilde{a}(n_{i+1} - 1) = \frac{3}{2}n_{i+1} - 3,$$

$$(6.4) \quad \tilde{a}(n_{i+1}) = 2n_{i+1} - 4.$$

Note that, in every step from (6.2) up to (6.3) we add 1 simultaneously to values of the arguments and of the right hand sides. Thus in the fundamental point $m_{i+1} = n_{i+1} - 1$ we have

$$n_i + l_h + x = n_{i+1} - 1$$

and

$$2n_i + 2l_h - 4 + x = \frac{3}{2}n_{i+1} - 3$$

such that

$$(6.5) \quad n_{i+1} = 2n_i + 2l_h - 4.$$

Now we should prove that the numbers

$$\frac{m_{i+1} - 3}{2} \mp 1 = \frac{n_{i+1} - 4}{2} \mp 1$$

i.e.

$$n_i + l_h - 5, \quad n_i + l_h - 3$$

are twin primes.

We have

$$\begin{aligned}
& \tilde{a}(n_i + l_h + t) = 2n_i + 2l_h - 4 + t, \\
(6.6) \quad & \tilde{a}(n_i + l_h + t + 1) = 2n_i + 2l_h - 3 + t,
\end{aligned}$$

where $0 \leq t \leq n_i + l_h - 7$. Distinguish two case.

1) Let l_h be even. Then, for even values of t the numbers $n_i + l_h + t + 1$ are odd and from equalities (6.6) we have

$$\gcd(n_i + l_h + t + 1, 2n_i + 2l_h - 4 + t) = 1.$$

or

$$\gcd(n_i + l_h + t + 1, n_i + l_h - 2 + t/2) = 1$$

and

$$\gcd(t/2 + 3, n_i + l_h - 5) = 1, \quad 0 \leq t/2 \leq (n_i + l_h - 7)/2.$$

Thus $n_i + l_h - 5$ is prime.

On the other hand, for odd values of t , taking into account that $n_i + l_h + t + 1$ is even, from equalities (6.6) we have

$$\gcd(n_i + l_h + t - 1, 2n_i + 2l_h - 4 + t) = 1,$$

$$\gcd(2n_i + 2l_h + 2t - 2, 2n_i + 2l_h - 4 + t) = 1$$

and

$$\gcd(t + 2, n_i + l_h - 3) = 1, \quad 0 \leq t \leq n_i + l_h - 7, \quad t \equiv 1 \pmod{2}.$$

Thus $n_i + l_h - 3$ is prime as well and the numbers $n_i + l_h - 5$, $n_i + l_h - 3$ are indeed twin primes.

2) Let l_h be odd. Then, using again equalities (6.6), by the same way, we show that the numbers $n_i + l_h - 5$, $n_i + l_h - 3$ are twin primes.

Besides, note that $n_i + l_h - 4 \equiv 0 \pmod{6}$ and, thus $m_{i+1} = n_{i+1} - 1 = 2n_i + 2l_h - 5 \equiv 3 \pmod{12}$. This completes the induction. ■

7. ALGORITHM WITHOUT TRIVIAL INCREMENTS

Sequences of the considered type in this paper and in [4] contain too many points of trivial 1-increments. For example, 10000 terms of sequence $\{\tilde{a}(n)\}$ give only 8 pairs of twin primes. Therefore, the following problem is actual from the computation point of view just as from the research point of view : to accelerate this algorithm for receiving of twin primes by the omitting of the trivial increments. Below we solve this problem.

Lemma 1. *If sequence $\{\tilde{a}(n)\}$ has a miner increment Δ in even point, then Δ is prime.*

Proof. Let even N be a point of a miner increment and $M = N - k$ be a point of the previous nontrivial increment. We distinguish two cases: M is even and M is odd.

a) Let M be even. Then we have

$$\tilde{a}(M) = 2M - 4,$$

$$\tilde{a}(M + 1) = 2M - 3,$$

...

$$\tilde{a}(M + k - 1) = 2M + k - 5,$$

$$(7.1) \quad \tilde{a}(N) = \tilde{a}(M + k) = 2M + 2k - 4,$$

where k is the least positive integer for which the point $M + k$ is the point of a nontrivial increment. We see that

$$\Delta = \Delta(N) = k + 1.$$

Since in this case k is even, then

$$\gcd(M + k - 2, 2M + k - 5) = d > 1$$

and, therefore,

$$\gcd(k + 1, M - 3) = d > 1.$$

Thus some prime divisor P of $M - 3$ divides $k + 1$ and, therefore, $k + 1 \geq P$. All the more,

$$k + 1 \geq p,$$

where p is the least prime divisor of $M - 3$. Since in the considered case $M - 3$ is odd, then p is odd. But, since $p - 2 \leq k - 1$, then in the run of formulas (7.1) there is the following

$$\tilde{a}(M + p - 2) = 2M + p - 6.$$

Nevertheless, the following value of argument is $M + p - 1 \equiv 0 \pmod{2}$ and both of the numbers $M + p - 3$ and $2M + p - 6$ are multiple of p . This means that $k \leq p - 1$, such that we have

$$\Delta = \Delta(N) = k + 1 = p.$$

2) M is odd. This case is considered quite analogously. Note that here $p \geq 2$. ■

Lemma 2. *Let sequence $\{\tilde{a}(n)\}$ have a minor increment Δ in odd point. If the sequence has the previous nontrivial increment in even point, then Δ is even such that $(\Delta + 4)/2$ is prime.*

Proof. Let odd N be a point of a minor increment and $M = N - k \equiv 0 \pmod{2}$ be a point of the previous nontrivial increment. Then we again have the run of formulas (7.1). Since here k is odd, then

$$\gcd(M + k, 2M + k - 5) = d > 1$$

and, therefore,

$$\gcd((k + 5)/2, M - 5) = d > 1$$

Thus some prime divisor P of $M - 5$ divides $(k + 5)/2$ and, therefore, $k + 5 \geq 2P$. All the more,

$$k + 5 \geq 2p,$$

where p is the least prime divisor of $M - 5$. Since in the considered case $M - 5$ is odd, then p is odd. But in the run of formulas (7.1) there is the following

$$\tilde{a}(M + 2p - 6) = 2M + 2p - 10.$$

Nevertheless, the following value of argument is $M + 2p - 5 \equiv 0 \pmod{1}$ and both of the numbers $M + 2p - 5$ and $2M + 2p - 10$ are multiple of p . This means that $k \leq 2p - 5$, such that we have

$$\Delta(N) = k + 1 = 2p - 4.$$

■

Quite analogously we obtain the following lemma.

Lemma 3. *Let sequence $\{\tilde{a}(n)\}$ have a miner increment Δ in odd point. If the sequence has the previous nontrivial increment in odd point, then Δ is odd such that $\Delta + 4$ is prime.*

Remark 1. *A little below we shall see that actually for nontrivial increments the conditions of Lemma 3 do not appear. But the proof of Lemma 3 plays its role!*

Note now that in proofs of Lemmas 1-3 p is always the least prime divisor of $M - 5$ or $M - 3$, where M is point of the "previous nontrivial increment," we obtain the following algorithm for the receiving of twin primes.

Theorem 5. *1) Let n_m be point of the m -th main increment of sequence $\{\tilde{a}(n)\}$ and P_m be the least prime divisor of the product $(n_m - 5)(n_m - 3)$. Then the first point N_1 of miner increment is*

$$(7.2) \quad N_1 = \begin{cases} n_m + P_m - 1, & \text{if } P_m | (n_m - 3), \\ n_m + 2P_m - 5, & \text{if } P_m | (n_m - 5). \end{cases}$$

2) Let N_i be a point of a miner increment of sequence $\{\tilde{a}(n)\}$ and p_i be the least prime divisor of the product $(N_i - 5)(N_i - 3)$. If N_i does not complete the run of points of the miner increments after n_m , then the following point of miner increment is

$$(7.3) \quad N_{i+1} = \begin{cases} N_i + p_i - 1, & \text{if } p_i = 2 \text{ or } p_i | (N_i - 3), \\ N_i + 2p_i - 5, & \text{if } p_i > 2 \text{ and } p_i | (N_i - 5). \end{cases}$$

3) If the point N_h completes the run of points of miner increments after n_m , then the following point of main increment is

$$(7.4) \quad n_{m+1} = 2N_h - 4.$$

Note that (7.4) corresponds to (6.5).

Corollary 4. *Conditions of Lemma 3 never satisfy.*

Proof. From (7.3) we conclude that after every odd point of miner increment follows even point of miner increment. ■

Remark 2. *In connection with Theorem 5 it is interesting to consider a close processes of receiving of twin primes. Let a be odd integer (positive or negative) and N_i be even. Let p_i be the least prime divisor of the product $(N_i - a - 2)(N_i - a)$ (in case of positive a , $N_i - a - 2 \geq 3$). Put*

$$N_{i+1} = N_i + p - 1.$$

One can conjecture that for some $j \geq i$, the numbers $N_j - a - 2$, $N_j - a$ will be twin primes. An important shortcoming of such process from the calculating point of view is the impossibility to use the formal algorithms for computation of the gcd.

8. PROPERTIES OF MINER INCREMENTS IN SUPPOSITION OF FINITENESS OF TWIN PRIMES

Condition 1. *There exists the maximal second of twin primes N_{tw} such that all seconds of twin primes belong to interval $[5, N_{tw}]$.*

Corollary 5. *There exists the last point n_T of a main increment of the sequence $\{\tilde{a}(n)\}$.*

Lemma 4. *If Condition 1 satisfies, then the set of the points righter n_T of nontrivial (miner) increments is infinite.*

Proof. Suppose that there exists the last point $n = \nu$ of a nontrivial increment, i.e. the set of points of miner the increments is not more than finite. Since we have

$$\tilde{a}(\nu) = 2\nu - 4,$$

then for every positive integer x , we find

$$\tilde{a}(\nu + x) = 2\nu - 4 + x.$$

In particular, for $x = \nu - 5$,

$$\tilde{a}(2\nu - 5) = 3\nu - 9.$$

But now the following point $2\nu - 4$ is a point of nontrivial increment. Indeed, $\gcd(2\nu - 6, 3\nu - 9) = \nu - 3$. Since, evidently, $2\nu - 4 > \nu$, then we have contradiction. ■

Besides, from the proof of Lemma 4 the following statement follows.

Lemma 5. *After every $n \geq n_T$ there is not a run of more than $n - 5$ trivial increments.*

Lemma 6. *Before every nontrivial increment of the magnitude t we have exactly $t - 2$ trivial increments.*

Proof. Indeed, by the run of formulas (6.2), on every segment

$$[n_i + l_j + 1, n_i + l_{j+1} - 1]$$

we have exactly $l_{j+1} - l_j - 1$ points of trivial increments and after that we obtain a nontrivial increment of the magnitude $l_{j+1} - l_j + 1$. ■

9. SOME ARITHMETICAL PROPERTIES OF POINTS OF THE MINER INCREMENTS OF SEQUENCE $\{\tilde{a}(n)\}$

Further we continue study sequence $\{\tilde{a}(n)\}$.

Lemma 7. *If M_i is an even point of miner increment, then M_i is not multiple of 3.*

Proof. We use induction. Since $n_m \equiv 1 \pmod{3}$, then, by (8.2), $p_0 > 3$ and it is easy to see that M_1 is not multiple of 3. Indeed, in (8.2) it is sufficient to consider cases $p_0 \equiv 1 \pmod{3}$ and $p_0 \equiv 2 \pmod{3}$. Further, using (8.1), note that if the case $M_i \equiv 1 \pmod{3}$ is valid, then the passage from M_i to M_{i+1} is considered as the passage from n_m to M_1 . If, finally, $M_i \equiv 2 \pmod{3}$, then $p_i = 3$, and again M_{i+1} is not multiple of 3. ■

Lemma 8. *If N_i is an odd point of miner increment, then the congruence $N_i \equiv 5 \pmod{6}$ is impossible.*

Proof. Since, by (7.3), after every odd point of miner increment t immediately follows the even point $t + 1$ of miner increment, then we should have $N_i + 1 \equiv 0 \pmod{6}$. This contradicts to Lemma 7. ■

Lemma 9. *If $N_i \equiv 4 \pmod{6}$ is a point of miner increment, then the magnitude of increment in point N_{i+1} is not less than 5.*

Proof. Since from Lemmas 7-8 we have $N_{i+1} - N_i \geq 3$, then the lemma follows from Lemma 6. ■

Lemma 10. *After every even point of miner increment N_i of the form $N_i \equiv 2 \pmod{6}$ follows the odd point $N_i + 1$ of miner increment (of the form $6l+3$).*

Proof. Since $N_i - 5 \equiv 0 \pmod{3}$, then by (7.3), in this case $p_i = 3$ and point $N_{i+1} = N_i + 2p_i - 5 = N_i + 1$ is the following increment. ■

Lemma 11. *The magnitude Δ of every miner increment either $\Delta = 2$ or $\Delta \geq 5$. Moreover, in the second case the previous miner increment has the form $6m + 4$.*

Proof. From Lemmas 7,8 all points of miner increments have one of the form $6t + i$, $i = 1, 2, 3, 4$. Besides, from (7.3) and Lemma 10 the miner increments $\Delta = 2$ occur after every points of miner increments of the form $6t + i$, $i = 1, 2, 3$, while, by Lemma 9, after every point of miner increments of the form $6t + 4$ we have a miner increment not less than 5. ■

Lemma 12. *If Condition 1 satisfies then there are infinitely many points of miner increment of the form $6m + 4$.*

Proof. In view of Lemmas 4 and 11, it is sufficient to prove that the process (7.3) which contains only $p = 2$ is finite. Let N_i be point of miner increment 2 such that all follow miner increments are 2. By Lemma 6, it is possible only if all points $N_i, N_i + 1, N_i + 2, \dots$ are points of miner increments. Consider any even point $N_j \equiv 1 \pmod{3}$, $j \geq i$. Since $N_j - 3$ and $N_j - 5$ are not multiple by 2 or 3, then, by (7.3), $N_{j+1} - N_j > 1$. This contradiction completes the proof. ■

10. A POSTULATE AND THE INFINITY OF TWIN PRIMES

Let S be the set of those even N for which $N - 3$ is not the first of a pair of twin primes.

Denote $p(N)$ ($q(N)$) the least prime divisor of $N - 1$ ($N - 3$). Denote A_1 (A_2) the set of those even $N \in S$ for which $p(N) < q(N)$ ($p(N) > q(N)$).

Postulate 1. *For every odd prime P , there exist even numbers $N_1 = N_1(P) \in A_1$, $N_2 = N_2(P) \in A_2$ such that $p(N_1) \geq P$, $q(N_2) \geq P$ and for the smallest such N_1, N_2 we have $\max(N_1, N_2) < (\min(N_1, N_2))^2$.*

In connection of the postulate, let us write interesting sequences of the smallest N_1, N_2 for $P = 3, 5, 7, 11, \dots$,

47, ..., such that $p(N_1) = q(N_2) = P$:

$(N_1(P)) : 10, 26, 50, 254, 170, 392, 362, 944, 842, 1892, 1370, 2420, 1850, 2210, \dots$

$(N_2(P)) : 12, 38, 80, 212, 224, 530, 440, 854, 1250, 1460, 1742, 2504, 2282, 3434, \dots$

We see that requirement of the postulate, at least in limits of the calculated terms, is very weak.

Theorem 6. *If the Postulate 1 is true, then there exist infinitely many twin primes.*

Proof. Accepting Condition 1 (Section 8), we obtain a contradiction. Consider a prime $P > N_{tw} + 1$. Using the postulate, consider minimal even numbers $N_1^* = N_1^*(P) \in A_1$, $N_2^* = N_2^*(P) \in A_2$ such that $p(N_1^*), q(N_2^*) \geq P$. Distinguish four cases.

Case 1) $p(N_1^*) \leq q(N_2^*)$, $N_1^* < N_2^*$. Note that if $N < N_2^*$ and $N \in A_2$, then, in view the minimality of N_2^* , the number $N_2^* - 3$ is not multiple of the smallest prime divisor of $N - 3$ (in the opposite case N_2^* loses its minimality). Analogously, if $N < N_1^*$ and $N \in A_1$, then, in view the minimality of N_1^* , the number $N_1^* - 1$ is not multiple of the smallest prime divisor of $N - 1$. All the more, $N_2^* - 1$ (having the smallest prime divisor more than the smallest prime divisor of $N_2^* - 3$ which is, by the condition, $q(N_2^*) \geq p(N_1^*)$) and $N_2^* - 3$ are not multiple of the smallest prime divisor of $N - 1$, if $N \leq N_1^*$ and $N \in A_1$. Nevertheless, if $N_1^* < N < N_2^*$, $N \in A_1$, then without the postulate we do not state that the smallest prime divisor of N does not divide $N_2^* - 3$ and $N_2^* - 1$. But, according to the postulate, we have $N_1^* > \sqrt{N_2^*}$. Thus, both of numbers $N_2^* - 3$, $N_2^* - 1$ are not multiple of the minimal prime divisors of all $N < \sqrt{N_2^*}$. Therefore, they are twin primes more than N_{tw} . It is a contradiction.

Case 2) $p(N_1^*) \geq q(N_2^*)$, $N_2^* < N_1^*$ is the symmetric case and is considered quite analogously. Here we obtain the pair of twin primes $N_1^* - 3$, $N_1^* - 1$ more than N_{tw} .

The following two cases are simpler and considered without the postulate's inequality.

Case 3) $p(N_1^*) \leq q(N_2^*)$ but $N_1^* > N_2^*$. In this case, evidently, none of $N < N_2^*$ divide $N_2^* - 3$ or $N_2^* - 1$. Thus they are twin primes.

Case 4) $p(N_1^*) \geq q(N_2^*)$ but $N_2^* > N_1^*$ is symmetric to Case 3, such that $N_1^* - 3$, $N_1^* - 1$ are twin primes. ■

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